

# On the Fundamental Relationship Determining the Capacity of Static and Mobile Wireless Networks

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**Abstract**—Studying the capacity of wireless multi-hop networks is an important problem and extensive research has been done in the area. In this letter, we sift through various capacity-impacting parameters and show that the capacity of both static and mobile networks is fundamentally determined by the average number of simultaneous transmissions, the link capacity and the average number of transmissions required to deliver a packet to its destination. We then use this result to explain and help to better understand existing results on the capacities of static networks, mobile networks and hybrid networks and the multicast capacity.

**Index Terms**—Capacity, mobile networks, wireless networks

## I. INTRODUCTION

Wireless multi-hop networks, in various forms, e.g. wireless sensor networks, underwater networks, vehicular networks, mesh networks and unmanned aerial vehicle formations, and under various names, e.g. ad-hoc networks, hybrid networks, delay tolerant networks and intermittently connected networks, are being increasingly used in military and civilian applications.

Studying the capacity of these networks is an important problem. Since the seminal work of Gupta and Kumar [1], extensive research has been done in the area. Particularly, it was shown in [1] that in an ad-hoc network with a total of  $n$  nodes uniformly and *i.i.d.* on an area of unit size and each node is capable of transmitting at  $W$  bits/s and using a fixed and identical transmission range, the achievable per-node throughput, when each node randomly and independently chooses another node in the network as its destination, is given by  $\lambda(n) = \Theta\left(\frac{W}{\sqrt{n \log n}}\right)$ . When the nodes are optimally and deterministically placed to maximize throughput, the achievable per-node throughput becomes  $\lambda(n) = \Theta\left(\frac{W}{\sqrt{n}}\right)$ . In [2], Franceschetti *et al.* considered the same random network as that in [1] except that nodes in the network are allowed to use two different transmission ranges. They showed that by having each source-destination pair transmitting using the “highway system”, formed by nodes using the smaller transmission range, the per-node throughput can also reach  $\lambda(n) = \Theta\left(\frac{1}{\sqrt{n}}\right)$  even when nodes are randomly deployed. The existence of such highway was established using the percolation theory [3]. In [4] Grossglauser and Tse showed that in mobile networks, by leveraging on the nodes’ mobility, a per-node throughput of  $\Theta(1)$  can be achieved at the expense

of unbounded delay. Their work [4] has sparked huge interest in studying the capacity-delay tradeoffs in mobile networks assuming various mobility models and the obtained results often vary greatly with the mobility models being considered, see [5] for an example. Further, there is also a significant amount of work studying the impact of infrastructure nodes [6] and multiple-access protocol [7] on capacity and the multicast capacity [8]. We refer readers to [9] for a more comprehensive review of related work.

In this letter, we sift through these capacity-impacting parameters, e.g. routing protocols, traffic distribution, mobility, presence of infrastructure nodes, multiple-access protocol and scheduling algorithm, and find the fundamental relationship determining the capacity of both static and mobile networks. Specifically, considering a very generic network setting, we show that the network capacity is fundamentally determined by the link capacity, the average number of simultaneous transmissions, and the average number of transmissions required to deliver a packet to its destination. We then show how to use the result to explain and better understand existing capacity results [1], [2], [4]–[8].

## II. CAPACITY OF STATIC AND MOBILE NETWORKS

In this section, we establish the main result on the network capacity. Specifically, consider a total of  $n$  nodes distributed in a bounded area  $A$ . These nodes may be either mobile or stationary. Packets are transmitted between a source and its destination via multiple intermediate relay nodes. Each node can be either a source, a relay, a destination or a mixture. Let  $V$  be the node set. Let  $v_i \in V$  be a source node and let  $b_{i,j}$  be the  $j^{th}$  bit transmitted from  $v_i$  to its destination. Let  $d(v_i, j)$  be the destination of  $b_{i,j}$ . For unicast transmission,  $d(v_i, j)$  represents a single destination; for multicast transmission,  $d(v_i, j)$  represents the set of all destinations of  $b_{i,j}$ . Let  $h_{i,j}$  be the number of transmissions required to deliver  $b_{i,j}$  to its destination (or all destination nodes in  $d(v_i, j)$ ). Let  $\tau_{i,j,l}$ ,  $1 \leq l \leq h_{i,j}$  be the time required to transmit  $b_{i,j}$  in the  $l^{th}$  transmission and assume that the transmitting node is active during the entire  $\tau_{i,j,l}$  interval. Let  $Y_t$  be the number of simultaneous transmissions in the network at time  $t$ . Let  $N_{i,T}$  be the number of bits transmitted by  $v_i$  and which reached their respective destination during a time interval  $[0, T]$ , with  $T$  being a large but arbitrary number. The network capacity, denoted by  $\eta(n)$ , is defined as:

$$\eta(n) \triangleq \lim_{T \rightarrow \infty} \frac{\sum_{i=1}^n N_{i,T}}{T} \quad (1)$$

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Note that the routing protocol used in the network plays an important role in determining  $\eta(n)$  and other parameters like  $h_{i,j}$  and  $Y_t$ . The validity of analytical results established in this section however does not depend on the particular routing protocol being used. Therefore we do not assume the use of a particular routing protocol in the network.

The average number of transmissions required to deliver a randomly chosen bit to its destination, denoted by  $k(n)$ , equals

$$k(n) = \lim_{T \rightarrow \infty} \frac{\sum_{i=1}^n \sum_{j=1}^{N_{i,T}} h_{i,j}}{\sum_{i=1}^n N_{i,T}} \quad (2)$$

When  $T$  is sufficiently large and the network is *stable*, the amount of traffic in transit is negligible compared with the amount of traffic that has already reached its destination. Therefore, the following relationship can be established:

$$\lim_{T \rightarrow \infty} \frac{\sum_{i=1}^n \sum_{j=1}^{N_{i,T}} \sum_{l=1}^{h_{i,j}} \tau_{i,j,l}}{\int_0^T Y_t dt} = 1 \quad (3)$$

A network is called stable if for any fixed  $n$ , assuming that each node has an infinite queue, the queue length in any intermediate relay node storing packets in transit does not grow towards infinity as  $T \rightarrow \infty$ .

Assuming that each node transmits at a fixed capacity  $W$ , then  $\tau_{i,j,l} = \frac{1}{W}$ . It can be shown that

$$\lim_{T \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^{N_{i,T}} \sum_{l=1}^{h_{i,j}} \tau_{i,j,l} = \frac{1}{W} \lim_{T \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^{N_{i,T}} h_{i,j} \quad (4)$$

Further, let

$$E(Y) \triangleq \lim_{T \rightarrow \infty} \frac{\int_0^T Y_t dt}{T} \quad (5)$$

where  $E(Y)$  has the meaning of being the average number of simultaneous transmissions in the network. It then follows from (1), (2), (3), (4) and (5) that

$$\eta(n) = \frac{E(Y)W}{k(n)} \quad (6)$$

*Remark 1.* The techniques used in obtaining Equation (3) and subsequently Equation (6) is based on first considering transmissions in the network on the individual node level by aggregating the transmissions at different nodes, i.e.  $\sum_{i=1}^n \sum_{j=1}^{N_{i,T}} \sum_{l=1}^{h_{i,j}} \tau_{i,j,l}$  and then evaluating transmissions in the network on the network level by considering the number of simultaneous transmissions in the entire network, viz.  $\int_0^T Y_t dt$ . Equation (6) can also be obtained using Little's formula in queueing theory [10].

Equation (6) is obtained under a very generic setting and is applicable for network of any size. It reveals that the network capacity is fundamentally determined by the average number of simultaneous transmissions  $E(Y)$ , the average number of transmissions required for reaching the destination  $k(n)$  and the link capacity  $W$ . The two parameters  $E(Y)$  and  $k(n)$  are often related. For example, in a network where each node transmits using a fixed transmission range  $r(n)$ , reducing  $r(n)$  (while keeping the network connected) will cause increases in both  $E(Y)$  and  $k(n)$  and the converse. On the other hand,  $E(Y)$  and  $k(n)$  also have their independent significance, and

can be optimized and studied independently of each other. For example, an optimally designed routing algorithm can distribute traffic evenly and avoid creating bottlenecks which helps to significantly increase  $E(Y)$  at the expense of slightly increased  $k(n)$ , compared with shortest-path routing. Further, observing that each transmission will “consume” a disk area of radius at least  $\frac{Cr(n)}{2}$  in the sense that two simultaneous active transmitters must be separated by an Euclidean distance of at least  $Cr(n)$ , where  $C > 1$  is a constant determined by the interference model [1], the problem of finding the maximum number of simultaneous transmissions, viz. an upper bound on  $E(Y)$ , can be converted into one that finds the maximum number of non-overlapping equal-radius circles that can be packed into  $A$  and then studied as a densest circle packing problem (see [11] for an example).  $E(Y)$  can also be studied as the transmission capacity of networks [12]. For unicast transmission,  $k(n)$  becomes the average number of hops between two randomly chosen source-destination pairs and has been studied extensively [13]. As will also be shown in Section III,  $E(Y)$  and  $k(n)$  can be optimized separately to maximize the network capacity.

We also note an important special case of (6): when the total number of source-destination pairs equals to  $m$  and each source-destination pair equally shares the network capacity, the throughput per source-destination pair, denoted by  $\lambda(n)$ , is given by

$$\lambda(n) = \frac{E(Y)W}{mk(n)} \quad (7)$$

The total number of possible source-destination pairs in the network equals to  $n(n-1)$  and if each node randomly and independently chooses another node in the network as its destination, as considered in [1], [2], [4]–[7],  $m = n$ .

### III. APPLICATIONS OF (6) TO EXISTING RESULTS

In this section, we use the result on network capacity established in (6) and (7) to explain and better understand existing results [1], [2], [4]–[8] in the area. Unless otherwise specified, we consider a network with  $n$  nodes uniformly and *i.i.d.* on a unit square  $A$  and each node is capable of transmitting at a fixed rate of  $W$  bits/s. A node chooses its destination randomly and independently of other nodes and the total number of source-destination pairs equals to  $n$ , viz.  $m = n$ . In some literature [2], [6], [8], a different network area is considered and their results are converted into a unit square and discussed.

#### A. Static ad-hoc networks

In [1], Gupta and Kumar first considered the network defined above and that each node transmits using a fixed and identical transmission range  $r(n)$ . Given the above setting, it is straightforward to establish that  $E(Y) = \Theta\left(\frac{1}{r^2(n)}\right)$  (as pointed out in Section II, each transmission consumes a disk area of radius  $\Theta(r(n))$ ) and  $k(n) = \Theta\left(\frac{1}{r(n)}\right)$ . Using (7) and noting that  $m = n$ , it can be shown that  $\lambda(n) = \Theta\left(\frac{W}{nr(n)}\right)$ , viz. a smaller transmission range will result in a larger throughput. The minimum transmission range

required for the network to be connected is well known to be  $r(n) = \Theta\left(\sqrt{\frac{\log n}{n}}\right)$ . Accordingly, the per-node throughput becomes  $\lambda(n) = \Theta\left(\frac{W}{\sqrt{n \log n}}\right)$ . By placing nodes optimally (e.g. on grid points) however, the transmission range required for a connected network reduces to  $r(n) = \Theta\left(\frac{1}{\sqrt{n}}\right)$ . Thus the per-node throughput becomes  $\lambda(n) = \Theta\left(\frac{W}{\sqrt{n}}\right)$ . Therefore the  $\frac{1}{\sqrt{\log n}}$  factor is the price in reduction of network capacity to pay for placing nodes randomly, instead of optimally.

In the networks considered by Franceschetti *et al.* [2], two transmission ranges are allowed, viz. a smaller transmission range of  $\Theta\left(\frac{1}{\sqrt{n}}\right)$  for nodes forming the highway and a larger transmission range of  $r(n) = \Theta\left(\sqrt{\frac{\log n}{n}}\right)$  for ordinary nodes. Most transmissions are through the highway using the smaller transmission range while the larger transmission range is only used for the last mile, i.e. between a source (or destination) and its nearest highway node. Therefore both  $E(Y)$  and  $k(n)$  are dominated by the smaller transmission range and accordingly  $E(Y) = \Theta(n)$  and  $k(n) = \Theta(\sqrt{n})$ . It then readily follows that  $\lambda(n) = \Theta\left(\frac{1}{\sqrt{n}}\right)$ .

Observing that in a large network, a much smaller transmission range is required to connect most nodes in the network (i.e. forming a giant component) whereas the larger transmission range of  $\Theta\left(\sqrt{\frac{\log n}{n}}\right)$  is only required to connect the few hard-to-reach nodes [14], a routing scheme can be designed, which achieves a per-node throughput of  $\lambda(n) = \Theta\left(\frac{1}{\sqrt{n}}\right)$  and does not have to use the highway system, such that a node uses smaller transmission ranges for most communications and only uses a larger transmission if the next-hop node cannot be reached when using smaller transmission ranges.

### B. Mobile ad-hoc networks

In the mobile ad-hoc networks considered in [4], nodes are mobile and the spatial distribution of nodes is stationary and ergodic with stationary distribution uniform on  $A$ . Moreover, the trajectories of different nodes are i.i.d. A two-hop relaying strategy is adopted. In the first stage, a source transmits a packet to its nearest neighbor (acting as a relay). As the source moves around, different packets are transmitted to different relay nodes. In the second stage, either the source or a relay transmits the packet to the destination when it is closest to the destination.

Obviously the two-hop relaying strategy helps to cap  $k(n)$  at 2. Compared with a one-hop strategy where a source is only allowed to transmit when it is close to its destination, the two-hop relaying strategy also helps to spread the traffic stream between a source-destination pair to a large number of intermediate relay nodes such that in steady state, the packets of every source node will be distributed across all the nodes in the network. This arrangement ensures that every node in the network will have packets buffered for every other node. Therefore a node always has a packet to send when a transmission opportunity is available. In this way,  $E(Y)$  is also maximized. Since the Euclidean distance between a

node and its nearest neighbor is  $\Theta\left(\frac{1}{\sqrt{n}}\right)$ , it follows that  $E(Y) = \Theta(n)$  [4]. As an easy consequence of (6) and (7),  $\lambda(n) = \Theta(1)$  and  $\eta(n) = \Theta(n)$ . Capacity of mobile ad-hoc networks assuming other mobility models and routing strategies [5] can also be obtained analogously.

Given the insight revealed in (6) and (7), it can be readily shown that in a network with a different traffic model than that in [4], viz. each node has an infinite stream of packets for every other node in the network, a one-hop strategy can also achieve a network capacity of  $\eta(n) = \Theta(n)$ . Therefore the insight revealed in (6) and (7) helps to design the optimum routing strategy for different scenarios of mobile ad-hoc networks.

### C. Multicast capacity

Now we consider the multicast capacity of a network using a similar setting as that in [8]. It is assumed that all nodes use the same transmission range  $r(n) = \Theta\left(\sqrt{\frac{\log n}{n}}\right)$ . Each node chooses a set of  $l-1$  points randomly and independently from  $A$  and multicast its data to the nearest node of each point. Further, it is assumed that the multicast transmission from each source follows the path of the Euclidean minimum spanning tree rooted at the source. Let  $\vartheta_i$  be the rate at which  $v_i$  send data to its destination nodes. The multicast capacity of the network is defined as  $\eta(n) = \sum_{v_i \in V} \vartheta_i$ , which is consistent with the definition in (1).

According to the analysis in [8], when  $l = O\left(\frac{n}{\log n}\right)$ , the number of transmissions required to reach all  $l-1$  multicast destinations is  $k(n) = \Theta\left(\frac{\sqrt{l}}{r(n)}\right)$ .  $E(Y)$  is mainly determined by the transmission range  $r(n)$  and is little affected by the change to multicast. Therefore,

$$\eta(n) = \Theta\left(\frac{1}{r^2(n)} \times \frac{r(n)}{\sqrt{l}} W\right) = \Theta\left(W \sqrt{\frac{\log n}{ln}}\right)$$

When  $l = \Omega\left(\frac{n}{\log n}\right)$ , the density of the multicast destination nodes becomes high enough such that the probability that a single transmission will deliver the data to more than one destination nodes becomes high. Consequently  $k(n) = \Theta\left(\frac{1}{r^2(n)}\right)$  (i.e. the number of transmissions required to cover the entire network) and  $\eta(n) = \Theta(W)$ .

### D. Hybrid networks

Now we consider the impact of infrastructure nodes. In addition to  $n$  ordinary nodes, a set of  $M$  infrastructure nodes are regularly or randomly placed on  $A$  where  $M < n$ . These infrastructure nodes act as relay nodes only and do not generate their own traffic. Following a similar setting as that in [6], it is assumed that the infrastructure nodes have the same transmission range  $r(n) = \Theta\left(\sqrt{\frac{\log n}{n}}\right)$  and bandwidth  $W$  when they communicate with the ordinary nodes and these infrastructure nodes are inter-connected via a backbone network with much higher bandwidth. Further, it is assumed that the routing algorithm has been optimized such that these infrastructure nodes do not become the bottleneck, which may

be possibly caused by a poorly designed routing algorithm diverting excessive amount of traffic to the infrastructure nodes.

First consider the case that  $M = o\left(\frac{1}{r^2(n)}\right) = o\left(\frac{n}{\log n}\right)$ . In this situation, the number of transmissions involving an infrastructure node as a transmitter or receiver is small and has little impact on  $E(Y)$ . Further, it can be shown that the average Euclidean distance between a randomly chosen pair of infrastructure nodes is  $\Theta(1)$  [15]. That is, a packet transmitted between two infrastructure nodes moves by an Euclidean distance of  $\Theta(1)$  whereas a packet transmitted by a pair of directly connected ordinary nodes moves by an Euclidean distance of  $\Theta(r(n))$ . Therefore a transmission between two infrastructure nodes is equivalent to  $\Theta\left(\frac{1}{r(n)}\right)$  transmissions between ordinary nodes and the equivalent average number of simultaneous ordinary node transmissions equals to  $\Theta\left(\left(\frac{1}{r^2(n)} - M\right) + \frac{M}{r(n)}\right) = \Theta\left(\frac{1}{r^2(n)} + \frac{M}{r(n)}\right)$ . It follows that

$$\eta(n) = \Theta\left(\frac{\left(\frac{1}{r^2(n)} + \frac{M}{r(n)}\right)W}{\frac{1}{r(n)}}\right) = \Theta\left(\left(\sqrt{\frac{n}{\log n}} + M\right)W\right)$$

Therefore when  $M = o\left(\sqrt{\frac{n}{\log n}}\right)$ , the infrastructure nodes have little impact on the order of  $\eta(n)$ ; when  $M = \Omega\left(\sqrt{\frac{n}{\log n}}\right)$ , the infrastructure nodes start to have dominant impact on the network capacity and the above equation reduces to  $\eta(n) = \Theta(MW)$ . Noting that the fundamental reason why infrastructure nodes improve capacity is that they help a pair of ordinary nodes separated by a large Euclidean distance to leapfrog some very long hops. Therefore the same result in the above equation can also be obtained by analyzing the reduction in  $k(n)$  directly. The analysis is albeit more complicated.

When  $M = \Omega\left(\frac{n}{\log n}\right)$ , the number of simultaneous active infrastructure nodes becomes limited by the transmission range. More specifically, only  $\Theta\left(\frac{1}{r^2(n)}\right) = \Theta\left(\frac{n}{\log n}\right)$  infrastructure nodes can be active simultaneously. Further, each ordinary node can access its nearest infrastructure node in  $\Theta(1)$  hops. Therefore  $\eta(n) = \Theta\left(\frac{nW}{\log n}\right)$ .

The above results are consistent with the results in [6].

However we further note that when  $M = \Omega\left(\frac{n}{\log n}\right)$ , a smaller transmission range of  $\Theta\left(\frac{1}{\sqrt{M}}\right)$  is sufficient for an ordinary node to reach its nearest infrastructure node and hence achieve connectivity. A smaller transmission range helps to maximize  $E(Y)$  while  $k(n) = \Theta(1)$ . Therefore the achievable network capacity using the smaller transmission range is  $\eta(n) = \Theta(MW) = \Omega\left(\frac{nW}{\log n}\right)$ , which is better than the result  $\eta(n) = \Theta\left(\frac{nW}{\log n}\right)$  in [6]. Moreover, different from the conclusion in [6] suggesting that when  $M = \Omega\left(\frac{n}{\log n}\right)$ , further investment in infrastructure nodes will not lead to improvement in capacity, our result suggests that even when  $M = \Omega\left(\frac{n}{\log n}\right)$ , capacity still keeps increasing linearly with  $M$ . This capacity improvement is achieved by reducing the transmission range with the increase in  $M$ .

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